Depinning transition at the upper critical dimension

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We study the effect of quenched random field disorder on a driven elastic interface close to the depinning transition at the upper critical dimension $d_c=4$ using the functional renormalization group. We have found that the displacement correlation function behaves with distance x as $(\ln x\Lambda_0)^{2/3}$ for large x. Slightly above the depinning transition the force-velocity characteristics are described by the equation $v\sim f|\ln f|^{2/9}$, while the correlation length behaves as $L_v\sim f^{-1/2}|\ln f|^{1/6}$, where $f=F/F_c-1$ is the reduced driving force.

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The driven viscous motion of an elastic interface in a medium with randomly distributed pinning disorder has attracted considerable theoretical interest during the last decade and is in a state of rapid development. The reason is that, on one hand elastic interfaces in a disordered medium exhibit the rich behavior of glassy systems and on the other hand it can serve as a model for many experimental systems, such as domain walls in magnetically or structurally ordered systems with impurities and interfaces between immiscible fluids in porous media. Other closely related problems are the motion of a vortex line in an impure superconductor and the motion of a dislocation line in a solid [1–7]. In all these systems the basic physical ingredients are identical: the elastic forces tend to keep the interface flat, whereas the disorder locally promotes the wandering. In the dynamics, this interplay between quenched disorder and elasticity leads to the complicated response of the interface to an externally applied force. At zero temperature, a driving force F exceeding certain threshold value F_c is required to set the elastic interface in steady motion. The depinning transition can be considered as a nonequilibrium dynamical critical phenomenon [1] where a system becomes extremely sensitive to small perturbation. Recently, significant progress has been made in understanding the depinning transition [4,5] (for recent studies see Refs. [6–8]). It has been shown that the functional renormalization group (FRG) gives an adequate description of the critical behavior at the depinning transition if one presumes to consider a singular renormalized random force correlator. The scaling analysis shows that the disorder effects dominate over elasticity in dimensions $d<4$, and therefore $d_c=4$ is the upper critical dimension of the problem. Below d_c the interface undergoes the depinning transition at a critical driving force F_c and slightly above the critical force F_c the average velocity of the interface behaves as

$$v\sim(F-F_c)^{\theta},\quad F>F_c, \tag{1}$$

where θ is the critical exponent. The roughness exponent characterizing the width w of the wandering interface at the depinning transition is defined by

$$w\sim L^{\zeta}, \tag{2}$$

where L is the linear size of the interface. The FRG analysis carried out in Ref. [4] enabled one to compute the critical force F_c, and the critical exponents θ and ζ to order $\varepsilon=4-d$. In the case of random force disorder it was found that $\theta=1-\varepsilon/9+O(\varepsilon^2)$ and $\zeta=\varepsilon/3$ [4,5]. It was suggested in Ref. [5] that at the depinning transition the critical exponents for random bond and random field (RF) disorder are the same. Initially, the result for the roughness exponent was expected to be exact for all $d=1,2,3$ [5], but more recently, the nonzero two-loop correction to ζ has been found [7].

The results of FRG analysis have been checked by intensive numerical studies using both direct simulation [9–12] and different cellular automata models [13–15], which are believed to belong to the same universality class. The computed values of critical exponents are in a good agreement with the predictions of FRG, at least for $d=1,2,3$. In the numerical works [12,16] the depinning transition was studied at the upper critical dimension. However, to our knowledge, no explicit consideration of the depinning transition at the upper critical dimension $d_c=4$ is available so far. Another motivation to consider the depinning transition at the upper critical dimension is that some experimental elastic systems, for example, systems with dispersive elastic constant such as moving geological faults arising from earthquakes [1], or systems with long-range Coulomb interaction, have the upper critical dimension $d_c=3$ or 2. One expects that these systems may show a behavior similar to the behavior of a simple model at $d_c=4$ [17].

It is well known [18] that at the upper critical dimension the power laws modify to logarithmic corrections. While at the upper critical dimension the one-loop RG consists in summing the main logarithms, the two-loop RG takes into account the subdominant logarithms [19]. Due to the fact that close to the depinning transition the main logarithms are leading, the results of the one-loop RG treatment are expected to be exact at the upper critical dimension for $F\to F_c$.

In this Brief Report we consider the motion of an elastic interface in a disordered medium and our main purpose is to describe the critical dynamics near depinning threshold for $d=d_c$ by using FRG method to one-loop order. The motion of a d-dimensional interface obeys the equation

$$\lambda\frac{\partial z(x,t)}{\partial t}=\gamma\nabla^2 z+F+g(x,z), \tag{3}$$

where λ is the friction coefficient (or the inverse mobility), γ is the stiffness constant, and F is the driving force density.

The quenched random force $g(x,z)$ is assumed to be Gaussian distributed with the zero mean and the correlator

$$\langle g(x,z)g(x',z')\rangle=\delta^{(d)}(x-x')\Delta(z-z'). \tag{4}$$

To make this model well defined, one has to introduce the cutoff Λ_0^{-1} in the $\delta^d(x)$ function at scales of order of the impurity separation or other microscopic scales. We restrict our consideration to the case of random field disorder when the correlator $\Delta(z)=\Delta(-z)$ is a monotonically decreasing function of z for $z>0$ and decays rapidly to zero over a finite distance.

In Ref. [4] the RG analysis of model (3), (4) was carried out using the technique of path integrals in the one-loop approximation. After integrating out fluctuations in the momentum shell $\Lambda_l<|k|<\Lambda_0$, the following RG flow equations have been obtained

$$\frac{d\ln\lambda}{dl}=-\frac{K_d}{\gamma^2\Lambda_l^{\varepsilon}}\int_0^{\infty}dt\,te^{-t}\Delta''(\tilde{v}t), \tag{5}$$

$$\frac{dF}{dl}=\frac{K_d}{\gamma\Lambda_l^{2-d}}\int_0^{\infty}dte^{-t}\Delta'(\tilde{v}t), \tag{6}$$

where v is the average velocity of the interface, $\Lambda_l=\Lambda_0e^{-l}$, $\tilde{v}=\lambda v/(\gamma\Lambda_l^2)$, and $K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(d/2)$. Due to the tilt symmetry the stiffness constant γ does not renormalize. In the limit $\tilde{v}\to 0$ the disorder correlator $\Delta(z)$ renormalizes as

$$\frac{d\Delta(z)}{dl}=-\frac{K_d}{\gamma^2\Lambda_l^{\varepsilon}}\frac{d^2}{dz^2}\left[\frac{1}{2}\Delta^2(z)-\Delta(z)\Delta(0)\right]. \tag{7}$$

RG equations (5) and (7) are the basis for computation of the force-velocity characteristics in the vicinity of the depinning transition. In the following we analyze Eqs. (5) and (7) at $d=4$, i.e., for $\varepsilon=0$. Before considering the general solution of the RG equations, we will analyze the flow equation for $\Delta''(0)$,

$$\frac{d\Delta''(0)}{dl}=-\frac{3K_4}{\gamma^2}[\Delta''(0)]^2. \tag{8}$$

From Eq. (8) it follows that as in the case $d<d_c$, the second derivative of the disorder correlator at origin $\Delta''(0)$ diverges at the finite length $l_c=-\gamma^2/[3K_4\Delta_0''(0)]$ for any initial condition $\Delta_0''(0)<0$. Thus one obtains the Larkin length $L_c=\Lambda_0^{-1}e^{l_c}$ at the upper critical dimension. The divergence of the curvature of $\Delta(z)$ implies the generation of a cusp singularity: $\Delta(z)$ becomes nonanalytical at the origin and acquires for $l>l_c$ a nonzero derivative $\Delta'(0^+)<0$. It was shown in Ref. [4] that the cusp generated during the renormalization determines the threshold force of the depinning transition. Therefore we expect that even at the upper critical dimension the interface is pinned for a small enough driving force.

We will now consider the depinning transition at the upper critical dimension. Although Eq. (8) does not have a sense beyond the Larkin scale, nevertheless, we can still use flow equation (7) for the renormalized correlator. In contrast to $d<d_c$, where the critical behavior at the depinning transition is obtained from the fixed-point solution of Eq. (7) corresponding to the condition $d\Delta^*(z)/dl=0$, the solution of Eq. (7) describing the behavior at the depinning transition at the upper critical dimension depends explicitly on l. To find the scaling form of the function $\Delta_l(z)$ at $d=4$ we look for an automodel solution of Eq. (7) in the form $\Delta_l(z)=K_4^{-1}\gamma^2\phi(l)\rho(\psi(l)z)$. Note that the latter reflects the scaling behavior at the depinning transition. Substituting this scaling ansatz into Eq. (7), we obtain the simultaneous equations for $\phi(l)$, $\psi(l)$, and $\rho(z)$:

$$\phi'(l)=-\phi^2(l)\psi^2(l),\quad \psi'(l)=-\phi(l)\psi^3(l), \tag{9}$$

$$[z\rho(z)]'=[\rho^2(z)/2-\rho(z)\rho(0)]''. \tag{10}$$

Equations (9) imply that $\psi(l)/\phi(l)=a$ is a constant that will be determined below. This condition allows us to find $\phi(l)=(3a^2l)^{-1/3}$ and $\psi(l)=(3l/a)^{-1/3}$, so that the automodel solution of Eq. (7) results in

$$\Delta_l(z)=K_4^{-1}\gamma^2(3a^2l)^{-1/3}\rho(z(3l/a)^{-1/3})). \tag{11}$$

Equation (11) is the pendant of the fixed-point solution of the disorder correlator at $d<4$. One should bear in mind that the FRG equation in this case ($d=d_c$) gives the exact large-scale behavior, while for $d<d_c$ one must rely on the ε expansion. The solution of Eq. (10) with the initial condition $\rho(0)=1$, which formally coincides with the equation for the fixed-point disorder correlator at $\epsilon=3$ [1], can be written as

$$\rho(z)-1-\ln\rho(z)=z^2/2. \tag{12}$$

Note that $\rho(z)$ has a cusp at origin so that its behavior near $z=0$ is given by $\rho(z)=1-|z|+\frac{1}{3}z^2+\cdots$. The constant a must be defined from the initial condition for the disorder correlator. Indeed, flow equation (7) for the disorder correlator implies that in the case of RF disorder the RF strength $c=\int_{-\infty}^{+\infty}\Delta_l dz$ is conserved to one-loop order [4] (it was shown in Refs. [7,8] that the above integral is not conserved in the two-loop order FRG), i.e., it does not depend on l. Therefore the constant a in ansatz (11) is determined by the strength c of the bare disorder correlator as $a\approx 1.55K_4^{-1}\gamma^2c^{-1}$, where we have used the integral $\int_{-\infty}^{+\infty}\rho(z)dz\approx 1.55$. To higher orders of FRG, the nonuniversal constant a is determined by higher moments of the bare disorder correlator. For the bare disorder correlator being a smooth function, the RG flow generates as in the case $d<d_c$ singularities on the scale l_c, which result in the cusp of the running disorder correlator, and therefore in the existence of the threshold force $F_c>0$. Because automodel solution (11) has the cusp on all scales, one should use it only beyond the Larkin scale. The full solution of flow equation (7), of course, depends on the initial condition, nevertheless, the latter is expected to approach solution (11) in the limit l

$\to\infty$. From Eq. (11) one can immediately derive the scaling relations for the first derivatives of the running disorder correlator,

$$\Delta_l(0)=K_4^{-1}\gamma^2(3a^2l)^{-1/3}, \tag{13}$$

$$\Delta_l'(0^+)=-K_4^{-1}\gamma^2(9al^2)^{-1/3}, \tag{14}$$

$$\Delta_l''(0)=2K_4^{-1}\gamma^2(9l)^{-1}. \tag{15}$$

Using the above results, we will now calculate the displacement correlation function $B(q)=\langle z_q z_{-q}\rangle$ that describes the roughness of the interface at the upper critical dimension. Simple scaling analysis shows that the correlation function satisfies the following flow equation [17]:

$$B(q)=e^{4l}B(qe^l;\Delta_l,F_l). \tag{16}$$

In order to extract the behavior for long-wavelength correlations at the depinning transition, $F=F_c$, we put $qe^l=\Lambda_0$ and expand Eq. (16) in powers of Δ. After some algebra this yields

$$B(q)=\frac{\Delta_l(0)}{\gamma^2q^4}=\frac{8\pi^2}{(3a^2)^{1/3}}\frac{1}{q^4[\ln\Lambda_0/q]^{1/3}}, \tag{17}$$

where in order to obtain the final expression on the right-hand side of Eq. (17) we have used Eq. (13). In a direct analogy with the case $d<4$, Eq. (17) holds simultaneously in the equilibrium and at the depinning threshold at least within the one-loop approximation [5,7]. The Fourier transform of Eq. (17) results in the following real-space displacement correlation function for large distance x:

$$B(x)\sim(\ln x\Lambda_0)^{2/3}. \tag{18}$$

To obtain the force-velocity characteristics we have to integrate the flow equations in the vicinity of F_c, i.e., in the limit of small $\tilde{v}$. Substituting Eq. (15) into flow equation (5) for the friction coefficient and Eq. (14) into the flow equation for driving force (6), we obtain

$$\frac{d\ln\lambda_l}{dl}=-\frac{2}{9l}, \tag{19}$$

$$\frac{dF_l}{dl}=-\frac{\gamma\Lambda_0^2}{(9a)^{1/3}}l^{-2/3}e^{-2l}. \tag{20}$$

Equations (19) and (20) describe the renormalization of λ_l and F_l to one-loop order beyond the Larkin scale at the upper critical dimension. In contrast to Eqs. (19) and (20) below the Larkin scale the friction coefficient λ_l increases under renormalization in accordance with Eq. (5), while the driving force is essentially not renormalized ($dF_l/dl\simeq0$). Integration of Eqs. (19) and (20) over l starting from l_c yields the following scaling relations for the friction coefficient and the driving force:

$$\lambda_l=\lambda_0(l/l_c)^{-2/9}, \tag{21}$$

$$F_l-F=-F_c\simeq-0.2\Lambda_0^2(\gamma c)^{1/3}, \tag{22}$$

where λ_0 is the friction coefficient on the scale l_c. In order to obtain the renormalized friction coefficient one should express l in Eq. (21) through the correlation length L_v according to $L_v=\Lambda_0^{-1}e^{l_v}$. In $d<d_c$ dimensions the relation between L_v and v reads, $L_v\sim v^{-1/(z-\zeta)}$, where $z=2-(\varepsilon-\zeta)/3$ is the dynamic exponent relating the time and the length scale [4,5]. Let us now derive the relation between L_v and v at the upper critical dimension. To do this we need first the relation between the time scale and the space scale, which is derived by using the relation $t\sim\lambda_l L^2/\gamma$ with λ_l given by Eq. (21) as $t\sim L^2(\ln L)^{-2/9}$. Following Ref. [4], the correlation length L_v can be derived by equating the systematic drift of the interface and the height fluctuation (or equivalently from the velocity-velocity correlation function at equal times [6]), which is the square root of Eq. (18), $vt\sim(\ln L)^{1/3}$. Combining the latter with the above relation between t and L gives the correlation length at the upper critical dimension as

$$L_v\sim v^{-1/2}|\ln v|^{5/18}. \tag{23}$$

The use of the relation, $F_{l_v}=v\lambda_{l_v}$, which is obtained by stopping the renormalization at l_v, where F_{l_v} is given by Eq. (22), λ_{l_v} is given by Eq. (21) with $l_v=\ln L_v\Lambda_0$, and $L_v\sim v^{-1/2}$ [the logarithmic correction to L_v in Eq. (23) results in higher order terms in the force-velocity characteristics] gives the implicit form of the force-velocity characteristics in the vicinity of the depinning transition at the upper critical dimension as

$$F-F_c\sim\frac{v}{|\ln v|^{2/9}}. \tag{24}$$

Within the one-loop consideration the interface velocity v under the logarithm of Eq. (24) can be replaced by $F-F_c$. Substituting v from Eq. (24) into Eq. (23) we express the correlation length as a function of the driving force as $L_v\sim f^{-1/2}|\ln f|^{1/6}$, where the reduced driving force $f=F/F_c-1$ is introduced.

We have checked that results (13)–(15), and (21) derived here for $d=4$ are consistent with the corresponding results of Ref. [4] for $d<4$, so that Eqs. (1) and (2) tend to Eqs. (24) and (18) for $d\to4$, respectively.

In the recent numerical study of the depinning transition of driven interfaces at the upper critical dimension in the random-field Ising model (RFIM) [20] the logarithmic corrections to the force-velocity characteristics were chosen in the form

$$[v(f)/f]^{1/\phi}\sim|\ln f|. \tag{25}$$

The best fit to numerical data was obtained with $\phi=0.40\pm0.09$. Taking into account that the numerical determination of the logarithmic corrections is difficult, this value is in a fair agreement with our exact result $\phi=2/9$. The reason of the discrepancy might be due to the fact that simulations are not carried out in the asymptotic regime, $\ln f \ll -1$. Indeed, the simulations in Ref. [20] were performed for $|\ln f| = 1$–4.5.

In the remaining part of this paper we will discuss the contributions of the subdominant logarithms, which appear in the two-loop order of RG. Using the results of Ref. [8] we find that the two-loop correction to disorder correlator (11) at $d=4$ has the form

$$\Delta_{2l}(z) = K_4^{-1}\gamma^2(9al^2)^{-2/3}\rho_2(z(3l/a)^{-1/3}), \tag{26}$$

where $\rho_2(z)$ obeys the following differential equation:

$$\{[1-\rho(z)][\rho_2(z)-\rho_2(0)] + 1/2\{[\rho(z)-1]\rho'^2(z)+\rho(z)\}\}'' + z\rho_2'(z) + 4\rho_2(z) = 0 \tag{27}$$

with $\rho(z)$ given by Eq. (12). Following Ref. [8], we impose the boundary condition $\rho_2(0)=0$ [$\rho_2(z)$ is the counterpart of the function $y_2(u)$ of Ref. [8], so that this condition is required for the consistency of the results for both $d<4$ and $d=4$]. Expanding $\rho_2(z)$ in a Taylor series we obtain $\rho_2(z) = -|z| + 19/18z^2 + \cdots$. The leading correction to Eq. (17) has the form $-\Delta_l'(0^+)^2/(\gamma q)^4 \int_k 1/k^2(k+q)^2$ [8], so that the logarithmic correction to roughness (18) behaves as $(\ln x\Lambda_0)^{-1/3}$ and, therefore, is irrelevant for large x. In order to take into account the two-loop corrections to force-velocity characteristics we need the two-loop correction to friction, which reads [8]

$$K_4^2/\gamma^4\{\Delta''(0^+)^2 + \Delta'''(0^+)\Delta'(0^+)[3/2-\ln 2]\}. \tag{28}$$

Substituting Eq. (26) into Eq. (5) and Eq. (11) into Eq. (28) we obtain the two-loop contribution to Eq. (19) as C/l^2, where $C = -[17/2+\ln 2]/54 \approx -0.17$. The two-loop correction to driving force (6) results only in a shift of the threshold force F_c, which is a nonuniversal quantity, so that we will not consider it. Using the same arguments that led us to Eqs. (23) and (24), we arrive at

$$f \sim l_v^{1/3}\exp(-2l_v), \quad v(f)/f \sim l_v^{2/9}(1+C/l_v). \tag{29}$$

Equations (29) express the force-velocity characteristics to two-loop order of FRG by using the parametric representation in terms of the correlation length $l_v = \ln L_v\Lambda_0$. The latter allows us to avoid the asymptotic expansion, which leads to a complicated expression [18].

In order to describe the crossover to the asymptotic behavior given by Eq. (24), one needs in addition to Eq. (29) the corrections to scaling. The latter are not available, since so far only the asymptotic solution of flow equations (5)–(7) is known. Due to these reasons the extension of the description of the critical behavior at the depinning transition to larger v is a nontrivial problem.

In conclusion, we have considered the effects of quenched random field disorder on the driven elastic interface at the upper critical dimension close to the depinning transition. We have shown that the interface undergoes the depinning transition at the critical driving force F_c, and we have obtained the logarithmic corrections to the displacement correlation function, the correlation length, and the force-velocity characteristics. In approaching the depinning transition, our one-loop results become exact. We hope that the analytical results derived here will be of interest for numerical studies of the depinning transition.

After this paper was finished, we learned that the more precise estimation of ϕ from the simulations of RFIM [21] is in a very good agreement with our prediction 2/9.

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